TITLE: Shear Reduction of 2D Point Vortex Diffusion

Shear Reduction of 2D Point Vortex Diffusion

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Abstract.
Theory and simulations establish the effects of shear on the collisional diffusion of a 2D point vortex gas. For finite shear, the diffusion is considerably smaller than previous zero-shear theories predict, scaling inversely with the shear. Surprisingly, changing the sign of the applied shear changes the diffusion by an order of magnitude.

INTRODUCTION

The collisional diffusion of point vortices in 2 dimensions is a classic problem in non-equilibrium statistical physics, with relevance to turbulence and transport in Euler fluids and neutral plasmas, rotating superfluid helium, the behavior of Type-II superconductors, and dislocations in solids. Here, we analyze the effects of an overall shear in the fluid flow on the diffusion of the point vortices, finding that the diffusion is strongly reduced.

Early work on diffusion of 2D point vortices focused on the case of a quiescent, homogeneous shear-free gas [1, 2]. When the vortices are distributed randomly, representing high-temperature thermal fluctuations, Taylor and McNamara showed that the diffusion coefficient (for diffusion in 1 direction) has the following simple form:

$$D^{TM} = \frac{1}{8\pi}\sqrt{\sum_{\alpha}\frac{N_\alpha \gamma_\alpha^2}{\pi}} \qquad (1)$$

where N_α is the number of point vortices of type α, each with circulation γ_α. The diffusion coefficient is not an intensive quantity because the diffusion process is dominated by large "Dawson-Okuda vortices" whose size is of order the system size [2]. However, for finite temperature, these authors all suggested that Debye shielding limits the maximal vortex size to approximately the Debye length λ_D.

Here, we show that in the presence of applied shear, the Dawson-Okuda vortices are disrupted and the diffusive transport is greatly reduced [3, 4] compared to Eq. (1). This result may be relevant to current experiments and theories in fusion plasmas, which also observe reduced transport in the presence of shear [5]. In such plasmas, the fluctuations are unstable and turbulent; and so the transport is difficult to determine theoretically. However, in a stable gas of point vortices, statistical theory determines the transport explicitly. Our theory may therefore be considered as a simple paradigm for the shear-reduction of transport seen in more complex turbulent systems.

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The theory also applies directly to experiments on pure ion plasmas or pure electron plasmas confined in cylindrical columns by a uniform magnetic field $B\hat{z}$ [6, 7]. In some parameter regimes, the individual particles act as z-averaged "rods" of charge that $\mathbf{E}\times\mathbf{B}$ drift in (r,θ) due to the electric fields of all the other rods. Under such conditions, these $\mathbf{E}\times\mathbf{B}$ drifts are the main cause of collisional diffusion, dominating over the classical diffusion [8] caused by velocity-scattering collisions. Using laser "tagging" techniques, ion experiments have directly measured the test-particle diffusion, and show quantitative correspondence with the present theory [9]. Electron experiments in the 2D regime have established that the bulk viscosity is strongly enhanced as the shear is reduced [10], but the theory of this (closely related) transport coefficient is not yet complete.

The $\mathbf{E}\times\mathbf{B}$ dynamics of a collection of charged rods is isomorphic to a gas of point vortices [11]: each rod, with charge $\bar{q}\equiv q/L_p$ per unit length, is equivalent to a point vortex with circulation

$$\gamma\equiv-\bar{q}\,(4\pi c/B)\,. \tag{2}$$

Furthermore, such plasmas rotate with a rotation frequency $\omega(r)=v_\theta(r)/r$ that may have substantial radial shear, depending on the particle density $n(r)$. We characterize the shear by a local shear rate

$$S(r)\equiv r\,d\omega/dr. \tag{3}$$

For comparison to electron columns of length L_p, we take $\bar{q}=-e/L_p$ and $B>0$, giving $\gamma>0$ and $\omega>0$. For comparison to ion experiments, the sense of rotation remains positive if one takes $\bar{q}=+e/L_p$ and $B<0$, giving $\gamma>0$ and $\omega>0$. We also note that the 2D areal density $n[\mathrm{cm}^{-2}]$ considered here is related to the 3D density $n^{3\mathrm{D}}$ $[\mathrm{cm}^{-3}]$ as $n=L_pn^{3\mathrm{D}}$.

THEORY

In order to evaluate the diffusion in such a sheared plasma/point vortex gas, we consider N identical vortices confined to a cylindrical patch of radius R, with *uniform* 2D areal density

$$n=N/\pi R^2\,, \tag{4}$$

giving an average interparticle spacing $a\equiv(\pi n)^{-1/2}$. To this vortex patch an *external* sheared rotation $\omega(r)$ is applied, with uniform shear rate S.

In physical systems, the rotation $\omega(r)$ would follow from $n(r)$ through Poisson's equation. For this system a dimensionless measure of the shear rate can be defined as

$$s=2S/n\gamma\,, \tag{5}$$

which is the shear rate compared to local vorticity density. For a uniform patch, one sees that $n\gamma=2\omega$, but for non-uniform $n(r)$ the local vorticity density $n\gamma$ is distinct from the local rotation rate ω.

In this analysis, we focus on the case of moderate to strong dimensionless shear, i.e. $|s|>1$, although comparison to experiments suggests that the results are valid for

$|s| < 1$. We derive the results of our statistical theory for self diffusion, and compare these results to molecular dynamics and vortex-in-cell simulations. For comparision to ion experiments, see the paper by Anderegg *et al.* in this Proceedings [9]. We find that the simulations agree with our theory provided that s is negative (i.e. negative shear, the usual circumstance in a stable pure electron plasma).

Surprisingly, however, when $s > 0$, the transport observed in the simulations is roughly an order of magnitude smaller than our theory predicts. We will discuss a qualitative explanation of the vortex trapping effects which cause the differences in diffusion, but at present no precise theory exists. We note that our point vortices (with $\gamma > 0$) are *prograde* for $s > 0$, and *retrograde* for the experimentally relevant case of $s < 0$. Similar trapping effects cause macroscopic prograde vortices to move up (or down) a background vorticity gradient at a rate substantially less than would a retrograde vortex [12]. The connection between the microscopic diffusion and macroscopic gradient-driven dynamics becomes more apparent when the present diffusion theory is generalized to two species: when one species represents macroscopic vortices with large circulation, a non-diffusive (indeed anti-diffusive) gradient-driven velocity is obtained [13].

The theory analyzes the collisions of vortices moving with a background sheared flow, as shown schematically in Fig. 1. We first describe theory for the diffusion that applies to the negative shear regime $s \lesssim -1$. In this case, two separate collisional processes are responsible for radial diffusion: small impact parameter collisions between vortices, described by a Boltzmann formalism, and large impact parameter collisions, described by a quasilinear formalism.

In Fig. 1a, the shaded region shows the range of possible streamlines (in the center-of-mass frame) for small impact parameter collisions. For example, vortex 1 may flow up and to the right and then back down because of a collision with vortex 2, which correspondingly flows down, to the left, and back up. These binary collisions cause a net radial displacement of each vortex, and a sequence of uncorrelated collisions gives diffusion. The unshaded region shows possible large impact parameter collisions, where the vortices merely move slightly in and back out while streaming past each other. In this case, simultaneous interaction with other distant vortices is required for radial displacement and diffusion.

For these collisions, small and large impact parameters mean initial radial displacements between vortices that are smaller or larger than a distance $2l$, where the "trapping distance" l is defined as

$$l \equiv \sqrt{-\gamma/4\pi S} = a/\sqrt{-2s}\,. \tag{6}$$

Note that for $s > 0$, the trapping distance is undefined, since the trapping extends to infinity in this simple model; and this will be seen to have a profound effect on the diffusion.

The streamlines show the trajectories of 2 identical vortices in the shear flow, at radial positions r_1 and r_2. For simplicity, we take $|r_1 - r_2| \ll r_1$, and introduce local cartesian coordinates in a moving frame with an origin initially at $\mathbf{R}(0) = [\mathbf{r}_1(0) + \mathbf{r}_2(0)]/2$, and moving with the local fluid rotation velocity $\omega(R(0))$. The x axis of this frame corresponds to the radial direction, and the y axis corresponds to the direction of local

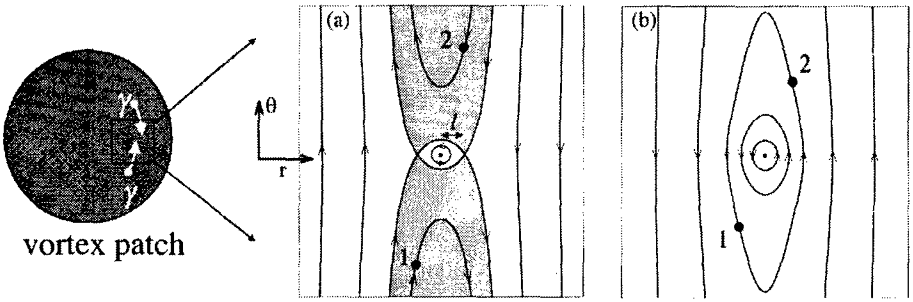

FIGURE 1. Collisional interaction of 2 point vortices, showing streamlines in the rotating frame for the interaction region for (a) $s < 0$, giving retrograde vortices; and (b) $s > 0$, giving prograde vortices.

flow (the θ-direction). In this coordinate frame, the vortices have positions $\Delta\mathbf{r} \equiv \mathbf{r} - \mathbf{R}$, where $\Delta\mathbf{r}_1 \equiv (x_1, y_1)$, and $\Delta\mathbf{r}_2 \equiv (x_2, y_2)$.

The 2-vortex interaction is described by the stream function

$$\psi(\Delta\mathbf{r}_1, \Delta\mathbf{r}_2) = \frac{S}{2}(x_1^2 + x_2^2) + \frac{\gamma}{4\pi}\ln[(x_1 - x_2)^2 + (y_1 - y_2)^2]. \tag{7}$$

Here the term proportional to the shear rate S is the stream function due to the shear flow, and the logarithmic term describes the vortex interaction. The motion of vortex 1 then follows from the Hamiltonian equations

$$\frac{dx_1}{dt} = -\frac{\partial\psi}{\partial y_1}, \quad \frac{dy_1}{dt} = \frac{\partial\psi}{\partial x_1}, \tag{8}$$

and similarly for vortex 2. Under this dynamics, ψ is a conserved quantity, and it is straightforward to show that $\Delta\mathbf{r}_1 + \Delta\mathbf{r}_2$ is also conserved, taking the value

$$\Delta\mathbf{r}_1 + \Delta\mathbf{r}_2 = 0. \tag{9}$$

Applying Eq. (9) to Eq. (7), ψ can be written as a function of the position of vortex 1 alone:

$$\psi(x_1, y_1) = S\left[x_1^2 - l^2\ln[4(x_1^2 + y_1^2]\right], \tag{10}$$

where the trapping distance l is given by Eq. (6). Contours of constant ψ are displayed in Fig. 1, with the origin $(x_1, y_1) = 0$ at the center of the figure.

For $s < 0$, vortices are retrograde (rotating against the shear), and they follow one of the symmetric pairs of trajectories shown in Fig. 1a. The vortex pairs always have reflection symmetry through the center of the figure, moving in opposite directions in this frame of reference. The separatrix in Fig. 1(a) has stagnation points at $x = \pm l$, $y = 0$, and using Eq. (10) this implies that the separatrix $x_s(y)$ is determined by

$$\begin{aligned} x_s^2 &= l^2(1 + \ln[(x_s^2 + y^2)/l^2] \\ &\approx l^2(1 + \ln[y^2/l^2]), \end{aligned} \tag{11}$$

where the second line is correct for $y \gg l$.

Binary Boltzmann Collisions

Vortices that begin in the shaded region inside the separatrix take a radial step due to their interaction. For inter-particle spacing greater than the trapping length, i.e. $a > l$ and $|s| > 1$, an uncorrelated sequence of these small impact parameter collisions will occur, causing radial diffusion of the vortices.

This diffusion can be estimated as $\nu\Delta r^2$, where the radial step Δr is of order l, and the collision rate $\nu \sim n|S|l^2$ is the number of vortices per unit time carried by the shear flow into the shaded trapping region of a given vortex (Fig. 1a). This results in a diffusion coefficient $\nu\Delta r^2 \sim n|S|l^4 \sim \gamma/|s|$.

A rigorous Boltzmann calculation of the radial diffusion due to these small impact parameter collisions agrees with this estimate. To determine the diffusion coefficient D^B, we integrate over random steps due to a flux Γ_y of vortices carried by the shear flow:

$$\Gamma_y = n|S\rho|\,, \tag{12}$$

where n is the areal density (in cm^{-2}) and $\rho = x_2 - x_1$ is the x-displacement (impact parameter) between the vortices when the vortices are well-separated in y (i.e. before the collision begins, but after the previous collision with some other vortex has ended). Let us call this y-displacement y_0. Then Fig. 1(a) shows that if $|\rho| \leq 2x_s(y_0)$, the two vortices will take a step in the x-direction of magnitude $|\rho|$ as they exchange x-positions in the collision.

The Boltzmann diffusion coefficient is therefore

$$D^B = \frac{1}{2}\int_{-2x_s(y_0)}^{2x_s(y_0)} d\rho\, \rho^2 \Gamma_y. \tag{13}$$

Using our previous expression for Γ_y, the integral can be performed yielding $D^B = 4n|S|x_s^4(y_0)$. Finally, Eq. (11) implies that $x_s(y_0)$ depends only logarithmically on y_0, so an estimate for y_0 is sufficient. Therefore, for y_0 we use the mean y displacement between collision events (the mean free path), $y_0 \simeq 1/(4ln)$. Then the Boltzmann diffusion coefficient becomes

$$D^B = \frac{\gamma}{2\pi^2|s|}\ln^2\left(\frac{e\pi^2 s^2}{4}\right), \tag{14}$$

where we have employed Eqs. (5), (6) and (11). The logarithm increases D_B over our previous estimate of $\gamma/|s|$ because the shaded region in Fig. 1(a) diverges logarithmically with increasing $|y|$, increasing both the size of the radial step Δr and the collision rate ν.

Multiple Distant Collisions

However, the Boltzmann result for self-diffusion given by Eq. (14) neglects diffusion from large impact parameters. In the Boltzmann description, two vortices with a large

impact parameter (outside the shaded region) stream by one another and suffer no net change in radial position. Actually, large impact parameter collisions are not isolated events; many are occuring simultaneously, leading to random motion in the fluctuations.

An estimate of the diffusion from these distant collisions is easily obtained. For these collisions the step size is now smaller than before, because interacting vortices are farther apart. If the impact parameter between vortices is ρ, the radial step Δr is of order $\Delta t\gamma/\rho$, where Δt is the time over which the interaction takes place. For particles streaming along unperturbed circular orbits, $\Delta t \sim 1/|S|$, which implies a small step $\Delta r \sim l^2/\rho$. There are many of these interactions per unit time; the collision rate is $\nu \sim n|S|\rho^2$, leading to a diffusion coefficient $\nu\Delta r^2 \sim n|S|l^4 \sim \gamma/|s|$, which is the same order as Eq. (14).

This diffusion from multiple distant collisions can be obtained more quantitatively from a quasilinear calculation [3] based on the Kubo formula

$$D^K = \int_0^\infty dt \langle \delta v_r(t) \delta v_r(0) \rangle. \tag{15}$$

Here $\delta v_r(t)$ is the radial velocity fluctuation of a test vortex at position (r,θ) due to $p = 1,...N$ other vortices at positions (r_p,θ_p), and $\langle\ \rangle$ denotes an ensemble average over a random distribution of the vortices within the vortex patch.

The velocity fluctuation is determined by a superposition of N flow fields [14],

$$\delta v_r(t) = -\frac{\gamma}{4\pi r} \sum_{p=1}^{N} \frac{\partial}{\partial\theta} \ln|\mathbf{r} - \mathbf{r}_p|^2, \tag{16}$$

easily written in terms of Fourier modes:

$$\delta v_r(t) = \frac{\gamma}{4\pi r} \sum_{p=1}^{N} \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{im}{|m|} e^{im(\theta-\theta_p)} (r_</r_>)^{|m|}, \tag{17}$$

where $r_{<(>)}$ is the lesser (greater) of r and r_p. The time integral in the Kubo formula can then be done using integration along unperturbed orbits, assuming that each vortex merely rotates about the center of the vortex patch, i.e. $r_p = \text{const}$, $\theta_p(t) = \omega(r_p)t + \theta_{p0}$. The ensemble average can also be easily calculated using standard techniques for random distributions, converting $\langle\sum_p\sum_{\bar{p}}\rangle$ to $\sum_{\bar{p}}\delta_{p\bar{p}}\int r_p dr_p d\theta_{p0} n(r_p)$.

The result, after performing the θ_{p0} and t integrals, is

$$D^K = \frac{\gamma^2}{(4\pi r)^2} \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} 2\pi^2 \int_0^\infty r_p dr_p n(r_p) \delta(m[\omega(r) - \omega(r_p)]) \left(\frac{r_<}{r_>}\right)^{2m}. \tag{18}$$

The δ-function, arising from the time integral over unperturbed orbits, implies that resonant interactions are the most important to the transport process. If we then assume that $\omega(r)$ is monotonic in r so that only $r = r_p$ contributes, the radial integral yields

$$D^K = \frac{n\gamma^2}{8r|\partial\omega/\partial r|} \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{1}{|m|}. \tag{19}$$

Collision Logarithms

The divergent sum occurs because nearby vortices following unperturbed orbits take a long time to separate and therefore take a large radial step. However, the sum can be cut off by noting that there is a minimum separation d for which unperturbed orbits are a good approximation. Adding the cutoff to Eq. (19) implies

$$D^K = \frac{\gamma}{2|s|} \ln[r/d], \tag{20}$$

where we have used Eq. (5).

One possible estimate for d is the trapping distance l, since vortices separated by l do not follow unperturbed orbits. Another possibility is that vortices diffuse apart before they are carried away by the shear, and so cannot be treated with unperturbed orbit theory. For vortices separated in r by a distance δ, the time to shear apart a distance of order δ is given by $1/|S|$, and the time to diffusively separate by a distance δ is $\delta^2/4D^K$. Setting the two times equal gives the diffusion-limited minimum separation $\delta = [4D^K/|S|]^{1/2}$.

Accordingly, in Eq. (20) we take the maximum of our two estimates:

$$d = \mathrm{Max}(\delta, l)\,. \tag{21}$$

Note that Eqs. (5), (6) and (20) imply that $\delta/l = \sqrt{(8\pi/|s|)\ln[r/d]}$, so the shear required to make $\delta < l$ is rather large. In our simulations it will turn out that $\delta > l$, so we use $d = \delta$ in Eq. (20).

Finally , the total diffusion coefficient is the sum of the Boltzmann and Kubo diffusion from small and large impact parameter collisions:

$$D = D^B + D^K\,. \tag{22}$$

Of course, Eq. (22) is correct only when the shear is large enough so that $D < D^{TM}$, where D^{TM} is the zero-shear result given by Eq. (1). However, comparing Eqs. (1) and (22), we see that only a small shear, $s \sim O(1/N^{1/2})$, is required to meet this inequality. When $|s| \approx 1$, Boltzmann collisions no longer occur since $a < l$, so $D^B = 0$; but Eq. (20) for D^K still holds. This implies that Eq. (22) is valid when $D^K \lesssim D^{TM}$, or $|s| \gtrsim \ln[r/d]\sqrt{16\pi^3/N}$. In other words, small shears wipe out the large-scale Dawson-Okuda vortices responsible for the diffusion predicted by Eq. (1).

SIMULATIONS

We have tested this theory using numerical simulations of N identical point vortices, initially placed randomly inside a circular patch, with an applied uniform external shear rate S. We find that Eq. (22) works well for $s \lesssim -1$, but overestimates the diffusion for $s \gtrsim 1$.

As a check of the numerics, we employ two separate simulation techniques, a 2D molecular dynamics (MD) method for point vortices, and a 2D particle in cell (PIC)

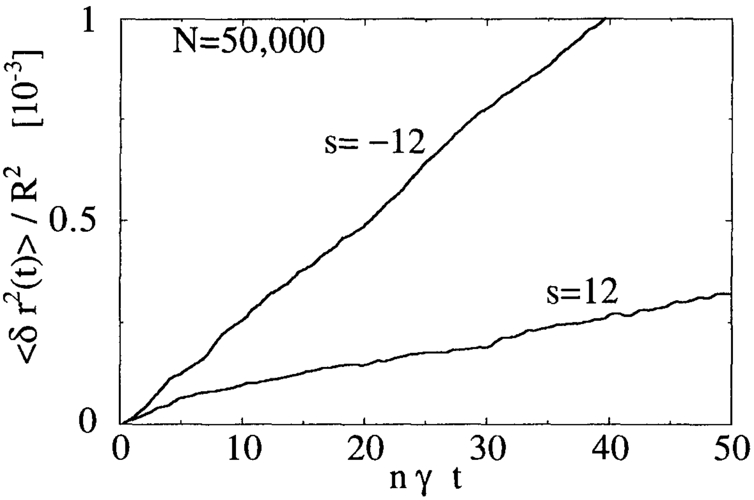

FIGURE 2. Mean square change in radial position of vortices vs. time, $\langle \delta r^2(t) \rangle$, for $N = 50,000$. Two shear rates are shown, $s = 12$ and $s = -12$.

simulation. The MD simulation is a standard N^2 code using the 4th order Runga-Kutta method. The PIC simulation has been described previously [12]. In the PIC simulation the diffusion coefficient is an increasing function of the number of grid points, but for sufficiently fine grid the diffusion is independent of the number of grid points to within our measurement error of about 30%. We use up to a 2048 × 2048 square grid in the largest PIC simulations. In both the PIC and MD codes, time steps are chosen to conserve energy at the 0.1% level or better over the course of the simulation, and angular momentum (mean square radius of the cylindrical patch) is typically conserved even more accurately. Also, the timestep was varied by factors of two in both codes, with no observable change in the diffusion.

In order to measure the diffusion coefficient, we chose as test particles all vortices in the band of radii from $0.43R$ to $0.57R$. For these vortices we followed the mean square change in radial position, $< \delta r^2(t) >$, where for vortex i, $\delta r_i(t) = r_i(t) - r_i(0)$. (We also verified that $< \delta r(t) >= 0$.)

Two examples with $N = 50,000$ are shown in Fig. 2, for the cases $s = \pm 12$. Here, the unexpectedly low diffusion for $s > 0$ is apparent. The diffusion coefficient is found from the equation $< \delta r^2(t) >= 2Dt$. More precisely, we fit a straight line to the segment of the curve that has nearly constant slope, and we take D as half the slope of that line.

RESULTS

Figures 3, 4, and 5 summarize our results for the self-diffusion coefficient. Figure 3 displays the diffusion coefficient as a function of the particle number N for four different values of the shear parameter, $s = 0$, -1.2, -12, and +12. The first case corresponds to

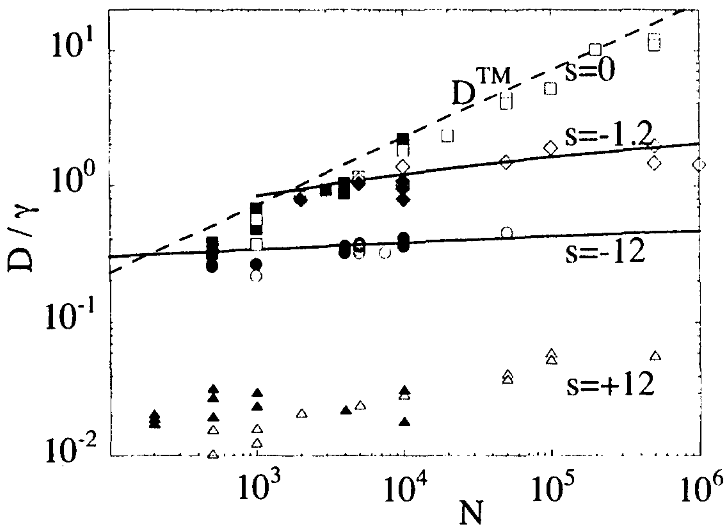

FIGURE 3. Self-diffusion coefficient vs. particle number N, for shear rates $s = 0$, -1.2, -12 and 12. Solid points are from MD simulations, open points are from PIC simulations. The dashed line is Eq. (1). The solid lines are Eq. (22), evaluated at $s = -1.2$ and $s = -12$.

a shear free plasma, and can be seen to match the expected Taylor-McNamara scaling, Eq. (1), shown by the dashed line. The cases $s = -1.2$ and -12 correspond to moderate and strong negative shear. The measured diffusion matches Eq. (22) quite well. However, for $s = +12$ the diffusion is an order of magnitude smaller.

For perspective, we note that both the Taylor-MacNamara diffusion and the collisional diffusion follow from the discreteness of the vorticity. If we let the discreteness go to zero by letting $N \longrightarrow \infty$ at *fixed total circulation*, then the diffusion will *decrease* as N increases. This is shown in Fig. 4, where the diffusion is scaled as $D/N\gamma$; this scaling can also be written $D/(4\pi cQ/B)$, where $Q = Nq$ is the total charge in the 2D system. From the perspective of Fig. 4, the shear-free diffusion $D^{TM}/N\gamma$ decreases as $N^{-1/2}$, since the fluctuation-induced Taylor-MacNamara vortices are weaker by $N^{1/2}$. On the other hand, for finite scaled shear s, the scaled diffusion $D/N\gamma$ decreases as N^{-1} since the collisionality is proportional to the discreteness.

In Fig. 5, the diffusion coefficient is shown as a function of shear for fixed particle number $N = 10^4$. The scaling with s matches Eq. (22) when $s < 0$. At large s, we obtain $D \propto 1/|s|$, although both Boltzmann and Kubo logarithms also introduce dependence on shear, which at low shear can be quite strong. Generally at low to moderate shear, the Kubo result dominates, but at low N or large shear the Boltzmann result dominates.

However, for $s > 0$ the vortices are prograde, and the simulations do not match Eq. (22): the measured diffusion is up to an order of magnitude less than the theory, depending on the shear rate. It is not surprising that the Boltzmann diffusion theory fails to work for this case; the Boltzmann picture of nearby particles reflecting off one

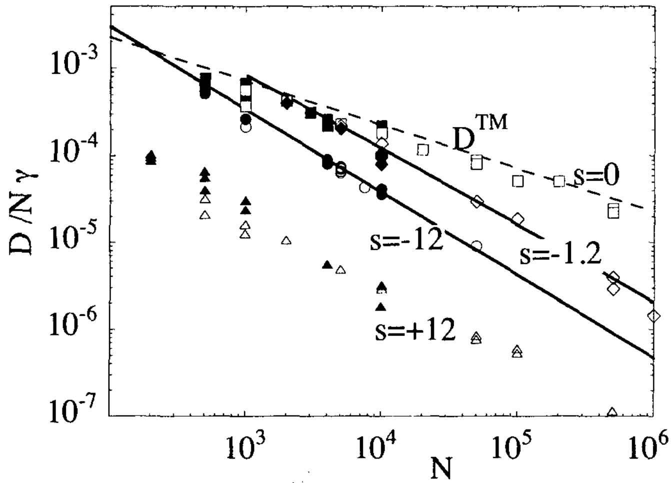

FIGURE 4. Same as Figure 3, except that the diffusion rates are scaled by the total circulation $N\gamma$. This figure shows that dividing a system with fixed circulation $N\gamma$ into more particles N (i.e. taking the "plasma limit") results in decreased collisional diffusion.

another is no longer correct. Two isolated vortices orbit around one another rather than suffer reflections, as shown in Fig. 1b. As a result, the measured $< \delta r^2(t) >$ has a large slope at early times as vortices begin to rotate around one another, but then relaxes to a smaller slope as vortices return to their initial radii (see Fig. 2).

Figure 5 shows that even if one neglects Boltzmann diffusion, Kubo diffusion by itself also overestimates the $s > 0$ simulation results. We believe that fluctuations now consist of several self-trapped vortices following elliptical orbits similar to the streamlines of Fig. 1b. Vortices return to their initial radii several times, and net transport occurs only through the break-up of these fluctuations through interaction with other similar fluctuations. The Kubo theory fails because the unperturbed orbit approximation fails for such fluctuations. A proper transport theory must go beyond the unperturbed orbit approximation in this case; such a theory will be the subject of future work.

DISCUSSION

This purely 2-dimensional theory needs further development with regard to Debye shielding, multiple species, and trapping effects in collisions of prograde vortices. Moreover, comparison to real 3-dimensional systems requires consideration of end effects (perturbations to the 2D velocity due to end confinement fields) and an understanding of the transition from 3D collisional dynamics to 2D bounce-averaged dynamics.

Considerations of Debye shelding by both Taylor and McNamara [1] and Dawson and Okuda [2] lead to the conclusion that thermal fluctuations will cause eddies with a maximal size of about λ_D. The more difficult problem of evaluating the dynamics of shielding during 2D point vortex collisions in a sheared plasma has yet to be solved. Preliminary analysis suggests that the shear disrupts the shielding, and a nonshielded

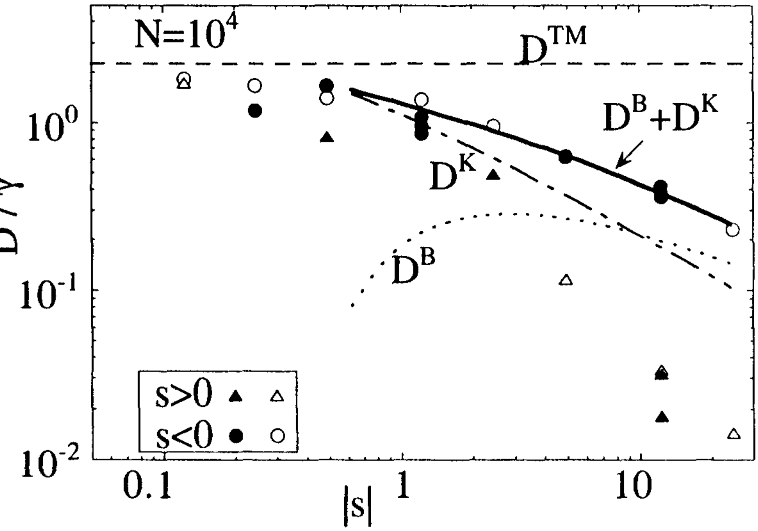

FIGURE 5. Self-diffusion coefficient vs. the shear rate s, for $N = 10{,}000$. Solid points are from MD simulations, open points are from PIC simulations. The dashed line is Eq. (1), and the solid line is Eq. (22). The dot-dashed line and the dotted line are the separate Kubo and Boltzmann contributions to the diffusion, Eqs. (20) and (14) respectively.

interaction may be a reasonable approximation. However, more work remains before this question can be answered definitively.

Extension of the present diffusion theory to multiple species is straightforward and is presently being developed [13]. Preliminary results show that there is a dissipative rearrangement of the vorticity of each species, and that species with higher circulation tend to concentrate in regions of higher total vorticity. This counter-intuitive, seemingly non-diffusive concentration effect is consistent with analyses [12] and experiments [15] which demonstrate that macroscopic vortices move *up* a background vorticity gradient. This effect is also consistent with the fundamental idea of an inverse cascade in 2D turbulence, whereby energy flows to large scales through the process of vortex merger [16].

Developing a consistent analysis of the trapping effects which occur with prograde vortices may represent the hardest of these 2D theory problems. In prograde shear, two like-sign point-vortices may be bound in compound "atoms" for arbitrarily long times; at present no theory can describe the effects of these bound states on diffusion. Similar bound states occur when one has both positive and negative vortices, as in neutral plasmas; then the problems include non-conservation of particle number and momentum, as well as infinite negative energies.

Perturbed 2D vortex velocities due to 3D end effects have been considered with regard to collisional viscosity [17, 18], and are presently being incorporated into diffusion theory [13]. Here, several effects may be important. The plasma (and charged rod) length L_p may vary with radial position. Moreover, the axial energy of the bounding particle

may affect the length of the bounce-averaged rod, and may also affect the $(r-\theta)$ motion of the rod. The lack of agreement between theory and experiments on 2D viscosity suggests that these effects are not yet adequately understood.

Finally, we note that experiments and theory are now developing the connections between 3-dimensional and 2-dimensional collisions in the presence of shear. Here, the controlling parameter seems to be the number of axial bounces which a particle makes before being sheared apart from neighboring particles, defined as $N_b \equiv (\bar{v}/2L_p)/r\omega_E' \equiv f_b/S$. The particle kinetic velocity $\bar{v}$ does not appear in present theory of 2D $\mathbf{E}\times\mathbf{B}$ drift collisions of charged rods, but it does appear in the analogous 3D theory of "long-range" or "$\mathbf{E}\times\mathbf{B}$ drift" collisions between charged particles. Indeed, theory and experiments [9] show that diffusion in the 2D regime is enhanced over diffusion in the 3D regime by the factor N_b, specifically $D^{3D}/D^{2D} = 0.47N_b$. Moreover, experiments on viscosity in the 2D regime [10] show essentially the same enhancement.

Thus, from the 2D perspective of this paper, one would say that vortex diffusion is reduced by shear, because the shear separates distant vortices before the collision can be completed. From the 3D perspective, one would say that diffusion is enhanced when the shear is sufficiently small that particles can have multiple correlated collisions as they bounce axially. Expanding this perspective to include end effects for both viscosity and diffusion remains the challenge for future theory and experiments.

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